COHOMOLOGY & BASE CHANGE II:

Recall:

Theorem (Grothendieck): $f: X \to Y$ proper of noetherian, $Y = \operatorname{Spec} A$, $\mathcal{F}$ coherent on X flat over Y. There exists a fin. complex $K^\bullet$ of fin. gen. proj. A-modules and isomorphism of functors, $H^p(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B) \xrightarrow{\sim} H^p(K^\bullet \otimes_A B)$ in category of A-alg.

Base change theorems:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

$g^*(R^p f_* \mathcal{F}) \longrightarrow R^p f'_*(g'^* \mathcal{F})$: By adjunction $g'^* \dashv g'_*$ we have $id \to g'_* \circ g'^*$ $\Rightarrow R^p f_* \longrightarrow R^p f_* g'_* g'^*$

(Grothendieck spectral sequence: $(R^p f_*) \circ (R^q g'_*)(A) = R^{p+q}(f_* \circ g_*)(A)$.

MISSING NOTES.

Corollary: $f: X \to Y$ as in theorem above (don't assume Y affine) and Y is reduced + connected. $\forall p$ TFAE,

(1) $Y \ni y \longmapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ where $X_y = X \times_Y \operatorname{Spec} k(y)$, $\mathcal{F}_y = \mathcal{F} \otimes_{\mathcal{O}_Y} k(y)$, is constant $\forall y$.

(2) $R^p f_* \mathcal{F}$ locally free and $R^p f_* \mathcal{F} \otimes_{\mathcal{O}_Y} k(y) \xrightarrow{\sim} H^p(X_y, \mathcal{F}_y)$.

If one of these conditions are satisfied, we also have $R^{p-1} f_* \mathcal{F} \otimes_{\mathcal{O}_Y} k(y) \xrightarrow{\sim} H^{p-1}(X_y, \mathcal{F}_y)$.

Proof: (2) $\Rightarrow$ (1) $R^p f_* \mathcal{F} = \mathcal{E}$, is locally free $\Rightarrow$ of constant rank r. Take $y \in Y$, $\mathcal{E} \otimes k(y) \cong H^p(X_y, \mathcal{F}_y)$ around $y \in Y$, $\mathcal{E}|_V \cong \mathcal{O}_Y^r|_V \Rightarrow \dim_{k(y)} H^p(X_y, \mathcal{F}_y) = \dim_{k(y)} \mathcal{E}|_V \otimes k(y) = r$.

Lemma: Y reduced, $\mathcal{F}$ coh. on Y $\dim_{k(y)} \mathcal{F} \otimes_{\mathcal{O}_Y} k(y) = r$ $\forall y \in Y \Rightarrow \mathcal{F}$ loc. free of rank r.

Proof: $Y = \operatorname{Spec} A$, let us take $y \in Y \Rightarrow \mathcal{F}_y$ fin.gen module by coherence and $\dim_{k(y)} \mathcal{F}_y \otimes k(y) = r$, we take generators $\sigma_1, \ldots, \sigma_r \in \mathcal{F}_y$ lifting to the generators of $\mathcal{F}_y \otimes k(y)$. $\sigma_1, \ldots, \sigma_r$ can be extended on a small neigh. V of y.

$\Rightarrow \exists \sigma: \mathcal{O}_Y^r|_V \to \mathcal{F}|_V$ and σ_y is surjective. $\Rightarrow \operatorname{Coker}(\sigma)_y = 0 \Rightarrow \exists$ some neigh. V st $\operatorname{Coker} \sigma|_V = 0 \Rightarrow \sigma$ is surjective (up to restricting). Want to show σ-inj., let $\mathcal{L} = \ker(\sigma)$. notice that

$\sigma_{y'} \otimes k(y')$: $\mathcal{O}_{Y,y'}^r \otimes k(y') \to \mathcal{F}_{y'} \otimes k(y')$ $\forall y' \in V$ is an isomorphism as surjection of vec. spaces of same (finite) dimension. We can conclude that $\mathcal{L}_{y'} \subseteq m_{y'} \mathcal{O}_{y'}^r$ inclusion $\forall y' \in V$. Recall that $V = \operatorname{Spec} B$ reduced $\Rightarrow B$ reduced ring. If y' corresponds to p-prime ideal, $?m_p \subseteq (pB_p) \cdot B_p^r$ $\forall p \in \operatorname{Spec} B \Rightarrow m_p = 0. \Rightarrow \mathcal{L}_{y'} = 0.$ $\square$

Lemma: Y reduced, noeth., affine, $\mathcal{F} \xrightarrow{\phi} \mathcal{L}$ morphism of loc. free coherent sheaves on Y. If $\dim(\operatorname{Im}(\phi \otimes k(y)))$ is locally const. $\Rightarrow \mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ with $\phi = \begin{bmatrix} 0 & \text{isom.} \\ 0 & 0 \end{bmatrix}$.

Proof: $\mathcal{F} \xrightarrow{\phi} \mathcal{L} \longrightarrow \mathcal{L}/\phi(\mathcal{F}) \to 0$, we want $\mathcal{L}/\phi(\mathcal{F})$ to be loc. free.

We know $\dim(\operatorname{Im}(\phi \otimes k(y)))$ is locally const. as $- \otimes k(y)$ is right exact,

$\mathcal{F} \otimes k(y) \longrightarrow \mathcal{L} \otimes k(y) \longrightarrow \mathcal{L}/\phi(\mathcal{F}) \otimes k(y) \to 0 \Rightarrow \mathcal{L}/\phi(\mathcal{F}) \otimes k(y) \cong \mathcal{L} \otimes k(y) / \operatorname{Im}(\phi \otimes k(y))$.

$\Rightarrow \mathcal{L}/\phi(\mathcal{F})$ is locally free.

Let $\Gamma(Y, \mathcal{F}) = M$, $\Gamma(Y, \mathcal{L}) = N$, $M \xrightarrow{\phi} N \longrightarrow N/\phi(M) \to 0$ and $N/\phi(M)$ projective. Consider the exact sequence $0 \to \phi(M) \hookrightarrow N \twoheadrightarrow N/\phi(M) \to 0$ and $N/\phi(M)$ is projective $\Rightarrow$ splits $\Rightarrow N = \phi(M) \oplus N/\phi(M)$. Also $\phi(M)$ is projective.

Thus $0 \to \ker \phi \hookrightarrow M \twoheadrightarrow \phi(M) \to 0$ also splits, $M \cong \ker \phi \oplus \phi(M)$, $N \cong \phi(M) \oplus N/\phi(M)$.

These are the sought out decompositions. $\square$

(1) $\Rightarrow$ (2) First pick $K^\bullet$ given by the theorem, we can assume $Y = \operatorname{Spec} A$ and $K^\bullet$ free complex. Let $d^p: K^p \to K^{p+1}$, $d^{p-1}: K^{p-1} \to \ker(d^p)$. Want to show, $\dim(\operatorname{Im}(d^p \otimes k(y)))$ is loc. constant

$\dim_{k(y)} H^p(X_y, \mathcal{F}_y) = \dim \ker(d^p \otimes k(y)) - \dim \operatorname{Im}(d^{p-1} \otimes k(y))$

$= \underbrace{\dim(K^p \otimes k(y))}_{\text{constant}} - \underbrace{\dim(\mathrm{Im}(d^p \otimes k(y))) - \dim(\mathrm{Im}(d^{p-1} \otimes k(y)))}_{\text{should be locally constant}}.$

$$\begin{array}{ccccc} K^{p-1} & \longrightarrow & K^p & \longrightarrow & K^{p+1} \\ \| & & \| & & \| \\ Z^{p-1} \oplus K'_{p-1} & \longrightarrow & B^p \oplus H_p \oplus K'_p & \longrightarrow & B^{p+1} \oplus K'_{p+1} \end{array}$$

iso   iso

, tensoring with $\otimes B$,
$H^p(K^\bullet \otimes_A B) = H^p \otimes B = H^p(K^\bullet) \otimes B$
$H^{p-1}(K^\bullet \otimes_A B) = H^{p-1}(K^\bullet) \otimes B$.

$R^p f_* \mathcal{F} \cong H^p(X, \mathcal{F})^\sim$, to use lemma 1 we want $\dim_{k(y)}(R^p f_* \mathcal{F} \otimes k(y)) = \dim_{k(y)}(H^p(X, \mathcal{F}) \otimes k(y))$
$= \dim(H^p(K^\bullet) \otimes k(y)) = \dim(H^p(K^\bullet \otimes k(y))) \Rightarrow$ use lemma 1. $\square$

Theorem: (Seesaw Thm): X complete variety $/k$ i.e. $X \times Y \to Y$ closed, T variety $/k$, $\mathcal{L}$ line bundle on $X \times T$, $T_1 = \{t \in T \cdot \mathcal{L}|_{X \times \{t\}}$ trivial on $X \times \{t\}\}$. $\Rightarrow T_1$ is closed in T also $\exists$ line bundle M on T_1 such that $p^* M = \mathcal{L}$ where $p: X \times T_1 \to T_1$.

Remark: If we have trivial line bundle M' on complete variety X,
$\Leftrightarrow h^0(X, M') > 0$ & $h^0(X, M'^{-1}) > 0$, note that X-complete $\Rightarrow H^0(X, \mathcal{O}_X) = k$.
and for $\Leftarrow$ take sections $\sigma \in H^0(X, M')$, $\delta \in H^0(X, M'^{-1}) \Rightarrow \sigma \otimes \delta = a \in H^0(X, \mathcal{O}_X)$ gives an iso $\mathcal{O}_X \cong M$.

Proof:
$X \times \{t\}$ is also complete $T_1 = \{t \in T, h^0(X \times \{t\}, \mathcal{L}|_{X \times \{t\}}) > 0\} \cap \{t \in T \mid h^0(X \times t, \mathcal{L}^{-1}|_{X \times \{t\}}) > 0\}$.
Recall: $y \longmapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ is upper semi cont $\Rightarrow t \mapsto \dim H^0(X \times \{t\}, \mathcal{L}|_{X \times \{t\}})$
$\Rightarrow T_1$ is closed, consider T_1 with reduced scheme structure.
We want to use corollary: $\dim_{k(y)} H^0(X \times \{t\}, \mathcal{L}|_{X \times \{t\}}) = \dim_{k(y)} H^0(X \times \{t\}, \mathcal{O}_{X \times \{t\}}) = 1$.
$\Rightarrow$ constant $\Rightarrow R^0 p_* \mathcal{L}$ locally free. (by lemma 1 it is a line bundle).

We want to show, $M = p_* \mathcal{L}$ i.e. $p^* p_* \mathcal{L} = \mathcal{L}$ by adjunction, $\exists p^* p_* \mathcal{L} \to \mathcal{L}$.

$$\begin{array}{ccc} X \times \{t\} & \xrightarrow{i} & X \times T \\ q \downarrow & \lrcorner & \downarrow p \\ \mathrm{Spec}\, k(y) & \xrightarrow{j} & X \end{array}$$

$\Rightarrow i^* p^* p_* \mathcal{L} \to \mathcal{L}|_{X \times \{t\}} = (pi)^* p_* \mathcal{L} \to \mathcal{L}|_{X \times \{t\}}$
$= (jq)^* (p_* \mathcal{L}) \longrightarrow \mathcal{L}|_{X \times \{t\}}$ and $(jq)^* p_* \mathcal{L} \cong q^* j^* p_* \mathcal{L}$
$\cong q^* q_* i^* \mathcal{L} \longrightarrow \mathcal{L}|_{X \times \{t\}} \Rightarrow q^* q_* \mathcal{L}|_{X \times \{t\}} \to \mathcal{L}|_{X \times \{t\}}$.

END OF LECTURE 9